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New classes of potentials for which the radial Schrödinger equation can be solved at zero energy

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Abstract
Given two spherically symmetric and short-range potentials V_0 and V_1 for which the radial Schrödinger equation can be solved explicitly at zero energy, we show how to construct a new potential V for which the radial equation can again be solved explicitly at zero energy. The new potential and its corresponding wavefunction are given explicitly in terms of V_0 and V_1, and their corresponding wavefunctions φ_0 and φ_1. V_0 must be such that it sustains no bound states (either repulsive, or attractive but weak). However, V_1 can sustain any (finite) number of bound states. The new potential V has the same number of bound states, by construction, but the corresponding (negative) energies are, of course, different. Once this is achieved, one can start then from V_0 and V, and construct a new potential $\overline{V}$ for which the radial equation is again solvable explicitly. And the process can be repeated indefinitely. We exhibit first the construction, and the proof of its validity, for regular short-range potentials, i.e. those for which $rV_0(r)$ and $rV_1(r)$ are L^1 at the origin. It is then seen that the construction extends automatically to potentials which are singular at $r = 0$. It can also be extended to V_0 long range (Coulomb, etc). We finally give several explicit examples.

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Dedicated to Professor Shinsho Oryu for his sixty fifth anniversary

1. Introduction

Consider the reduced radial Schrödinger equation for a spherically symmetric potential $V(r)$ [1]

$$\begin{cases} \varphi_\ell''(k,r) + k^2\varphi_\ell(k,r) = \left[\dfrac{\ell(\ell+1)}{r^2} + V(r)\right]\varphi_\ell(k,r), \\ r \in [0,\infty), \quad k \geqslant 0, \quad \varphi_\ell(k,0) = 0, \\ V(r) \text{ real, locally } L^1 \text{ for } r \neq 0, \quad \text{and } V(\infty) = 0. \end{cases} \tag{1}$$

We exclude, therefore, confining potentials like the harmonic oscillator, etc.

There are only a few potentials for which the radial Schrödinger equation can be solved explicitly for all k and all ℓ. These are essentially the square well, the Coulomb potential, sums of δ-function potentials [1]; and for potentials which are more singular than r^{-2} at the origin, but repulsive, only λ/r^4 [2], for which the solution is given in terms of very complicated Mathieu function. For confining potentials there is, of course, the case of harmonic oscillator for which one can solve the Schrödinger equation for all k and all ℓ.

If we restrict ourselves to the case of one single ℓ, then we can include the Bargmann potentials [1, 3], for which, by construction, the radial Schrödinger equation can be solved for one specific value of ℓ, and for all k. We remind the reader that the Bargmann potentials are those for which the S-matrix $S_\ell(k)$ is a meromorphic function of k in the k-plane. They can be constructed for each ℓ.

In the particular case of $\ell = 0$, the radial equation can be solved for all k for the following potentials [1]:

$$V_1(r) = \lambda\,\mathrm{e}^{-\mu r}, \tag{2}$$

$$V_2(r) = \frac{\lambda\,\mathrm{e}^{-\mu r}}{1 - \mathrm{e}^{-\mu r}} \quad \text{(Hulthén)}, \tag{3}$$

and, more generally, the Eckart potentials [4]:

$$V_3(r) = \frac{\lambda_1\,\mathrm{e}^{-\mu r}}{1 + C\,\mathrm{e}^{-\mu r}} + \frac{\lambda_2\,\mathrm{e}^{-\mu r}}{(1 + C\,\mathrm{e}^{-\mu r})^2}, \tag{4}$$

of which (3) is a particular case. The solutions are given in terms of hypergeometric functions.

In the case of $k = 0$, and for all ℓ, one can add the potential [5]:

$$V_4(r) = \frac{\lambda r^{\alpha-2}}{(C + r^\alpha)^2}, \qquad \alpha > 0, \quad \ell \geqslant 0. \tag{5}$$

An interesting particular case is when $\alpha = 2$:

$$V_5(r) = \frac{\lambda}{(C + r^2)^2}, \qquad \ell \geqslant 0. \tag{6}$$

Finally, for $k = 0$ and $\ell = 0$, one can solve the radial equation also for

$$V_6(r) = \frac{\lambda}{(C + r)^4}, \qquad \ell = 0. \tag{7}$$

We shall give later the explicit solutions for some of these potentials, when they are simple.

The purpose of the present paper is to show that if one can solve explicitly the radial Schrödinger equation at $k = 0$ for

$$\begin{cases} V_0(r) + \dfrac{\ell(\ell+1)}{r^2}, \\ \displaystyle\int_0^1 r|V_0(r)|\,\mathrm{d}r < \infty, \quad \int_1^\infty r^{2\ell+2}|V_0(r)|\,\mathrm{d}r < \infty, \\ \text{no bound states,} \end{cases} \tag{8}$$

and for $k = 0$ and $\ell = 0$ for the potential

$$\begin{cases} V_1(r),\ \ell = 0, \\ \displaystyle\int_0^1 r|V_1(r)|\,\mathrm{d}r < \infty, \quad \int_1^\infty r^2|V_1(r)|\,\mathrm{d}r < \infty, \\ \text{any number of bound states,} \end{cases} \tag{9}$$

then one can solve explicitly the radial equation, again at $k = 0$ and $\ell = 0$, for the potential

$$V(r) = V_0(r) + F_0(r)V_1(G_0(r)), \tag{10}$$

in terms of the solutions for (8) and (9). Here, the functions $F_0(r)$ and $G_0(r)$ are given very simply in terms of the solutions for (8).

To begin with, we consider the case $\ell = 0$. If we call by $\varphi_0(r)$ and $\chi_0(r)$ the two independent solutions for $V_0(r)$ defined by

$$\begin{cases} \varphi_0(0) = 0, \quad \varphi_0'(0) = 1; \quad \chi_0(0) = 1, \\ W(\varphi_0, \chi_0) = \varphi_0'\chi_0 - \varphi_0\chi_0' = 1 \end{cases} \tag{11}$$

$F_0(r)$ and $G_0(r)$ are given by

$$F_0(r) = [\chi_0(r)]^{-4}, \qquad G_0(r) = \frac{\varphi_0(r)}{\chi_0(r)}. \tag{12}$$

As we shall see later, since by assumption, $V_0(r)$ has no bound states, all the above quantities are meaningful because both $\varphi_0(r)$ and $\chi_0(r)$ defined by

$$\chi_0(r) = \varphi_0(r)\int_r^\infty \frac{\mathrm{d}t}{\varphi_0^2(t)}, \tag{13}$$

do not vanish anywhere, except for φ_0 at $r = 0$ (see details in the appendix). Note that the converse of (13) is

$$\varphi_0(r) = \chi_0(r)\int_0^r \frac{\mathrm{d}t}{\chi_0^2(t)}\,\mathrm{d}t. \tag{14}$$

The solution of the radial equation at $k = 0$ and $\ell = 0$ for the potential $V(r)$, (10), is then given by

$$\begin{cases} \varphi(r) = \chi_0(r)\varphi_1\left(\dfrac{\varphi_0(r)}{\chi_0(r)}\right), \\ \varphi(0) = 0, \end{cases} \tag{15}$$

where φ_1 is the regular solution of the radial equation for V_1, (9), defined by $\varphi_1(0) = 0$. By assumption, both φ_0 and φ_1 are known, and both vanish at $r = 0$ by definition. The above formula can be checked directly by differentiation. We shall see in the following section how it was found.

Remark 1. As shown in the appendix, $x = G_0(r)$ maps $r \in [0, \infty)$ into $x \in [0, \infty)$. The mapping is one to one, and is, of course, twice differentiable. Both $x = x(r)$ and its inverse $r = r(x)$ are increasing functions.

Once (15) is known, it is easy to generalize it to the case one has angular moment with $V_0(r)$ (see the appendix):

$$\begin{cases} V_0(r) + \dfrac{\ell(\ell+1)}{r^2}, \quad \ell \geqslant 0, \\ rV_0(r) \in L^1(0) \quad \text{and} \quad r^{2\ell+2}V_0(r) \in L^1(\infty), \\ \text{no bound states,} \end{cases} \tag{16}$$

$V_1(r)$ remaining unchanged. φ_0 and χ_0 are now the solutions of the radial equation with the potentials (16), so that (10) becomes

$$V(r) = \left[V_0(r) + \frac{\ell(\ell+1)}{r^2}\right] + F_0(r)V_1(G_0(r)). \tag{17}$$

All other formulae given above remain unchanged.

In short, if one can solve explicitly the radial equations

$$\begin{cases} \varphi_0''(r) = \left[V_0(r) + \dfrac{\ell(\ell+1)}{r^2}\right]\varphi_0(r), & \varphi_0(0) = 0, \\ \varphi_1''(r) = V_1(r)\varphi_1(r), & \varphi_1(0) = 0, \end{cases} \tag{18}$$

where V_0 and V_1 satisfy, respectively, the assumptions shown in (16) and (9), then the solution of

$$\varphi''(r) = \left[V_0(r) + \frac{\ell(\ell+1)}{r^2}\right]\varphi(r) + \frac{1}{\chi_0^4(r)}V_1\left(\frac{\varphi_0(r)}{\chi_0(r)}\right)\varphi(r) = V(r)\varphi(r), \tag{19}$$

with $\varphi(0) = 0$, is given by

$$\varphi(r) = \chi_0(r)\varphi_1\left(\frac{\varphi_0(r)}{\chi_0(r)}\right). \tag{20}$$

Remember that $\chi_0(r)$ is always defined by (13).

It is easy to check our assertion by differentiating twice φ, given by (20), and using (18). We shall see that one can replace $V_0(r)$ by strongly repulsive potentials which are more singular than r^{-2} at the origin. Examples will be provided for

$$V_0(r) = \frac{g}{r^n}, \qquad g > 0, \qquad n > 2, \tag{21}$$

for which the radial Schrödinger equation is soluble for all ℓ at $k = 0$ [6].

Remark 2 (The bound states)**.** As is well known, the nodal theorem [7] asserts that the number of bound states of $V(r)$, (10) or (19), is given by the number of the nodes of the regular wavefunction $\varphi(r)$, (20). Since neither $\chi_0(r)$ for $r \geqslant 0$, nor $\varphi_0(r)$ for $r > 0$, do not vanish (remember that, by assumption, V_0 has no bound states), and $x = \frac{\varphi_0(r)}{\chi_0(r)}$ maps $r \in [0, \infty)$ into $x \in [0, \infty)$ and the mapping is one to one, it is obvious on (20) that φ and φ_1 have the same number of nodes. Therefore, V_1, (9), and V, (10) or (19), have the same number of bound states. Of course, the energies of these states are different for V_1 and V. In any case, one has also the Bargmann bound for the number of bound states [1, 3, 4]:

$$n(V) = n(V_1) \leqslant \int_0^\infty r|V_1(r)|\,\mathrm{d}r < \infty. \tag{22}$$

It is easily seen that the Calogero–Cohn bound is invariant [4].

Remark 3 (Iterating the process)**.** Once we have the explicit solution (20) for equation (19), we can start now with the couple $[V_0(r), V(r)]$, instead of $[V_0(r), V_1(r)]$, and look for the solution of the radial equation at $k = 0$ for

$$\begin{cases} \overline{\varphi}''(r) = \overline{V}(r)\overline{\varphi}(r) \\ \overline{V}(r) = \left[V_0(r) + \dfrac{\ell(\ell+1)}{r^2}\right] + \dfrac{1}{\chi_0^4(r)}V\left[\dfrac{\varphi_0(r)}{\chi_0(r)}\right]. \end{cases} \tag{23}$$

We will find now, of course, the solution

$$\overline{\varphi}(r) = \chi_0(r)\varphi\left(\frac{\varphi_0(r)}{\chi_0(r)}\right), \tag{24}$$

where φ is given by (20). And this process can be continued as many times as we wish.

We end this introduction by giving one example with the potentials

$$\begin{cases} V_0(r) = \dfrac{\lambda a^2}{(1+ar)^4}, \qquad a > 0, \quad \lambda > 0, \quad \ell = 0, \\ \varphi_0(r) = \left(\dfrac{1+ar}{a\sqrt{\lambda}}\right) \sinh\left(\dfrac{\sqrt{\lambda}ar}{1+ar}\right), \\ \chi_0(r) = (1+ar)\left[\cosh\left(\dfrac{\sqrt{\lambda}ar}{1+ar}\right) - \dfrac{\cosh\sqrt{\lambda}}{\sinh\sqrt{\lambda}} \sinh\left(\dfrac{\sqrt{\lambda}ar}{1+ar}\right)\right] \end{cases} \tag{25}$$

and

$$\begin{cases} V_1(r) = \dfrac{gb^2}{(b^2+r^2)^2}, \quad g > 0, \quad b > 0, \\ \varphi_1(r) = \dfrac{(b^2+r^2)^{1/2}}{\sqrt{g-1}} \sinh\left(\sqrt{g-1}\,\mathrm{Arctg}\dfrac{r}{b}\right) \end{cases} \tag{26}$$

from which one can calculate φ by formula (20). Since the Schrödinger equation can be solved for the potential (26) for all ℓ, we can invert the roles of V_0 and V_1, and start with

$$V_0(r) + \frac{\ell(\ell+1)}{r^2} = \frac{gb^2}{(b^2+r^2)^2} + \frac{\ell(\ell+1)}{r^2}. \tag{27}$$

Here, for general ℓ, the solutions φ_0 and χ_0 are given in terms of hypergeometric functions [5]. We restrict ourselves to the case of $\ell = 0$, for which

$$\begin{cases} \varphi_0(r) = \dfrac{\sqrt{b^2+r^2}}{\sqrt{g-1}} \sinh\left(\sqrt{g-1}\,\mathrm{Arctg}\dfrac{r}{b}\right), \\ \chi_0(r) = \sqrt{b^2+r^2} \cosh\left(\sqrt{g-1}\,\mathrm{Arctg}\,\dfrac{r}{b}\right), \end{cases} \tag{28}$$

and take (25) as the second potential, with

$$\varphi_1(r) = \frac{1+ar}{a\sqrt{\lambda}} \sinh\left(\frac{\sqrt{\lambda}ar}{1+ar}\right). \tag{29}$$

We have tacitly assumed that we are looking at three dimensions, with $\ell = 0, 1, 2, \ldots$ in (1). One can, of course, choose any dimension $D \geqslant 2$. The reduced radial equation (1) reads now [10]

$$\begin{cases} \varphi''(r) + E\varphi(r) = \left[V(r) + \dfrac{L(L+1)}{r^2}\right]\varphi(r), \\ L = \ell + \dfrac{D-3}{2}, \quad \ell = 0, 1, \ldots. \end{cases} \tag{1$'$}$$

In fact, from Regge–Pole theory [1], we know that we can make ℓ continuous, or even complex. In all the examples we give later, one has to deal with hypergeometric functions of one kind or another, and there the dependence on ℓ is explicit.

For attractive potentials, there are many cases one has bound states at zero energy [1, 4]. Many instances of this are studied in [10]. For long-range attractive potentials, or for $D = 2$ and $\ell = 0$, because of the presence of $-1/4r^2$ in (1′), one may have even an infinite number of bound states. We refer the reader to [11] and [12] for detail studies of these, connected to the zero-energy radial Schrödinger equation.

2. Derivation of solution (15)

Consider the mapping $r \to x(r) = G_0(r) \equiv \varphi_0(r)/\chi_0(r)$, shown in (14). In the appendix, it is shown (see (A.12)–(A.15)) that this is a one to one mapping of $r \in [0, \infty)$ into $x \in [0, \infty)$, and is twice continuously differentiable. We can consider now the equation

$$\begin{cases} \varphi''(r) = [V_0(r) + V_1(r)]\,\varphi(r), \\ \varphi(0) = 0, \quad \varphi'(0) = 1, \end{cases} \tag{30}$$

where V_0 and V_1 satisfy the conditions shown in (8) and (9). We now make the change of variable and function

$$r \to x = \frac{\varphi_0(r)}{\chi_0(r)}, \qquad \psi(x) = \left.\frac{\varphi(r)}{\chi_0(r)}\right|_{r=r(x)}, \tag{31}$$

where $r(x)$ is the inverse mapping, i.e. the inverse function of $x = x(r)$. Obviously, $r(x)$ is also twice continuously differentiable. Differentiating now twice ψ with respect to x, and using (14), or (A.13), we easily find

$$\ddot{\psi}(x) = \left[\chi_0^4(r)V_1(r)\right]\big|_{r=r(x)}\,\psi(x). \tag{32}$$

There is no longer V_0 present. From the definition of $\psi(x)$, of $\varphi(r)$ given in (30), and (A.9), it is obvious that, because $\varphi(r) = r + o(r)$ as $r \to 0$, we have

$$\psi(0) = 0, \qquad \dot{\psi}(0) = \lim_{x\to 0}\dot{\psi}(x) = \lim_{r\to 0}[\varphi'(r)\chi_0(r) - \varphi(r)\chi_0'(r)] = 1. \tag{33}$$

Suppose now that, from the beginning, $V_1(r)$ in (30) was of the form:

$$\frac{1}{\chi_0^4(r)}V_1\left(x = \frac{\varphi_0(r)}{\chi_0(r)}\right), \tag{34}$$

where $xV_1(x) \in L^1$ at $x = 0$, and $x^2V_1(x) \in L^1$ at $x = \infty$. Then (32) would become

$$\begin{cases} \ddot{\psi}(x) = V_1(x)\psi(x), \\ \psi(0) = 0, \quad \dot{\psi}(0) = 0, \end{cases} \tag{35}$$

which we assume to be explicitly solvable. Then, from the definition (31), we would have for the solution of (30), with a V_1 of the form (34):

$$\varphi(r) = \chi_0(r)\psi\left(\frac{\varphi_0(r)}{\chi_0(r)}\right). \tag{36}$$

Combining all these, with a slightly different notation, we have therefore the following:

Theorem 1. *The solution of*

$$\begin{cases} \varphi''(r) = \left[V_0(r) + \dfrac{1}{\chi_0^4(r)}V_1\left(\dfrac{\varphi_0(r)}{\chi_0(r)}\right)\right]\varphi(r) \\ \varphi(0) = 0, \quad \varphi'(0) = 1, \end{cases} \tag{37}$$

where φ_0 and χ_0 are the two solutions of

$$\begin{cases} \varphi_0''(r) = V_0(r)\varphi_0(r), \\ \varphi_0(0) = 0, \quad \varphi_0'(0) = 1, \quad \text{no bound states}, \\ \chi_0(r) \text{ defined by (13)}, \quad \chi_0(0) = 1, \end{cases} \tag{38}$$

is given by

$$\varphi(r) = \chi_0(r)\varphi_1\left(\frac{\varphi_0(r)}{\chi_0(r)}\right), \tag{39}$$

where $\varphi_1(x)$ is the (regular) solution of

$$\begin{cases} \ddot{\varphi}_1(x) = V_1(x)\varphi_1(x), \\ \varphi_1(0) = 0, \quad \dot{\varphi}_1(0) = 1. \end{cases} \tag{40}$$

Therefore, if the Schrödinger equation at $k = 0$ and $\ell = 0$ can be explicitly solved for V_0 and V_1, then the solution of (37) is of the form (39). As we said in the introduction, one can check directly, that (39) is indeed the solution of (37). For bound states in (40) and (37), see below, after remark 4.

The above theorem can be generalized easily to higher ℓ by considering (16) and (18). The details of the properties of the corresponding φ_0 and χ_0 are given in the appendix. Following the same reasoning as for theorem 1, we find

Theorem 2. *Theorem 1 is valid if we add $\ell(\ell+1)/r^2$ to V_0 in (37) and (38), i.e.*

$$\begin{cases} \varphi''(r) = \left[\left(V_0(r) + \dfrac{\ell(\ell+1)}{r^2}\right) + \dfrac{1}{\chi_0^4(r)} V_1\left(\dfrac{\varphi_0(r)}{\chi_0(r)}\right)\right]\varphi(r), \\ \varphi(r \to 0) = \dfrac{r^{\ell+1}}{(2\ell+1)!!} + \cdots \end{cases} \tag{41}$$

and

$$\begin{cases} \varphi_0''(r) = \left[V_0(r) + \dfrac{\ell(\ell+1)}{r^2}\right]\varphi_0(r), \quad r^{2\ell+2}V_0(r) \in L^1(\infty), \\ \varphi_0(r \to 0) = \dfrac{r^{\ell+1}}{(2\ell+1)!!} + \cdots, \quad \text{no bound states}, \end{cases} \tag{42}$$

$\chi_0(r)$ defined by (13), $\chi_0(r \to 0) = \frac{(2\ell-1)!!}{r^\ell}$, while (40) remains unchanged. The solution is again provided by (39). This can also be checked directly.

Remark 4. Since the behaviour of $\varphi_1(x)$ is $x + \cdots$ as $x \to 0$, and $\varphi_0(r)/\chi_0(r) = r^{2\ell+1}$ as $r \to 0$, it is obvious in (39) that we have, as we should,

$$\varphi(r) = \alpha_\ell r^{\ell+1} + \cdots, \qquad r \to 0. \tag{43}$$

Likewise, it is easy to find

$$\varphi(r) = \beta_\ell r^{\ell+1} + \gamma_\ell r^{-\ell}, \qquad r \to \infty. \tag{44}$$

2.1. Bound states

So far, we have assumed that there are no bound states in (40). If there are n bound states with $V_1(x)$, i.e. in (40), which is the same for theorems 1 and 2, this means that φ_1 has n nodes for $x > 0$. And we have also n nodes for the full solution φ, given always by (39). The potentials V_1 and V have the same number of bound states. But, of course, the binding energies are different.

Remark 5. As we said in remark 3, once we have $V(r)$ and $\varphi(r)$, we can now start again with V_0 and V instead of V_0 and V_1, and proceed as before. This process can be repeated as many times as we wish, and we get more and more potentials for which the radial Schrödinger equation at zero energy can be solved. Also, the number of bound states, if any, remains the same. Unfortunately, the potentials and the wave functions become quickly very complicated. However, one may ask what will happen at the limit of infinite repetitions. This question is certainly not easy to answer.

2.2. Singular potentials

As we said in the introduction, the radial equation can be solved at $k = 0$ and for all ℓ for singular potentials which are just inverse powers potentials shown in (21). The solutions are given in terms of the modified Bessel and Hankel functions. We shall see explicit examples in the following section. We should mention here that, contrary to the case of regular potentials at the origin, i.e. those for which $rV_0(r) \in L^1$ at $r = 0$, here, because of strong singularities at $r = 0$, we find [1, 6] $\varphi_0(0) = \varphi_0'(0) = \cdots \varphi_0^{(n)}(0) = \cdots = 0$. All the derivatives of φ_0 vanish at $r = 0$. The normalization is therefore arbitrary, and cannot be made at $r = 0$. Once this is chosen (usually by the behaviour of φ at $r = \infty$), then $\chi_0(r)$ is given again by (13), and we have the Wronskian $W(\varphi_0, \chi_0) = 1$. Usually, in such a case, it is customary to normalize φ_0 at infinity, according to

$$\varphi_0(r) = r + C + o(1), \qquad r \to \infty, \tag{45}$$

which entails also

$$\chi_0(r) = 1 + o(1), \qquad r \to \infty. \tag{46}$$

As we shall see on explicit examples in the following section, our procedure for generating new potentials can go on without modifications.

3. Examples

3.1. Regular potentials

We have already given, in the introduction, as examples for the applications of theorems 1 and 2, the solutions of the Schrödinger equation for $\ell = 0$, $V_0 = (7)$, $V_1 = (6)$, or $V_0 = (6)$ together with $\ell(\ell+1)/r^2$, and $V_1 = (7)$. They are given by formulae (25)–(29). More examples are obtained by combining any two potentials among those given by (2)–(7). We need only the solutions φ and χ, the latter defined by (13), for these potentials.

(a) *Exponential potential*

$$\begin{cases} V(r) = \lambda\, \mathrm{e}^{-\mu r}, \quad \lambda > 0, \quad \mu > 0, \quad \ell = 0 \\ \varphi(r) = \alpha I_0\left(\dfrac{2\sqrt{\lambda}}{\mu}\, \mathrm{e}^{-\mu r/2}\right) + \beta K_0\left(\dfrac{2\sqrt{\lambda}}{\mu}\, \mathrm{e}^{-\mu r/2}\right), \\ \chi(r) = \dfrac{I_0\left(\frac{2\sqrt{\lambda}}{\mu}\, \mathrm{e}^{-\mu r/2}\right)}{I_0\left(\frac{2\sqrt{\lambda}}{\mu}\right)}, \quad \chi(0) = 1, \quad \chi(\infty) = \dfrac{1}{I_0\left(\frac{2\sqrt{\lambda}}{\mu}\right)}. \end{cases} \tag{47}$$

I_0 and K_0 are the modified Bessel and Hankel functions of order zero, and the constants α and β are determined to have $\varphi(0) = 0$ and $\varphi'(0) = 1$. It is then easy to show that, according to our general analysis of section 2, we have

$$\varphi(r \to \infty) = I_0\left(\frac{2\sqrt{\lambda}}{\mu}\right) r + \cdots. \tag{48}$$

The presence of r is due to the presence of $\log z$ in $K_0(z)$ when $z \to 0$ [13].

(b) *The potential (5) for* $\alpha > 0$, $\ell \geqslant 0$. The solutions φ and χ are given by combinations of hypergeometric functions of appropriate arguments. We refer the reader to [5] for details.

(c) *Hulthén potential* (3), $\ell = 0$. Here also, the solutions φ and χ are given in terms of appropriate hypergeometric functions F. See [1, 4] for details.

3.2. Singular potentials [6]

Here, we consider only three cases.

(d) *Inverse power potentials*, $\ell \geqslant 0$:

$$\begin{cases} V(r) = \dfrac{g}{r^n}, \quad g > 0, \quad n > 2\ell + 3, \\ \varphi(r) = r^{1/2} K_{\frac{2\ell+1}{n-2}} \left(\dfrac{2\sqrt{g}}{(n-2) r^{\frac{n-2}{2}}} \right), \\ \chi(r) = r^{1/2} \left[\alpha K_{\frac{2\ell+1}{n-2}} \left(\dfrac{2\sqrt{g}}{(n-2) r^{\frac{n-2}{2}}} \right) + \beta I_{\frac{2\ell+1}{n-2}} \left(\dfrac{2\sqrt{g}}{(n-2) r^{\frac{n-2}{2}}} \right) \right], \end{cases} \tag{49}$$

where I_ν and K_ν are the modified Bessel and Hankel functions. We must choose $n > 2\ell+3$ in order to comply with (A.23) [6].

The parameters α and β are determined for having $\chi(r)$ to comply with the asymptotic behaviours deduced from (13), for $r \to 0$ and $r \to \infty$. One has to remember the Wronskian

$$W[r^{1/2} K_\nu(\beta r^\sigma), r^{1/2} I_\nu(\beta r^\sigma)] = \sigma. \tag{50}$$

The case $\ell = 0, n = 4$ is particularly simple. One finds

$$\begin{cases} V(r) = \dfrac{g}{r^4}, \quad g > 0, \quad \ell = 0, \\ \varphi(r) = r\, \mathrm{e}^{-\sqrt{g}/r} \underset{r\to\infty}{=} r - \sqrt{g} + \cdots \\ \chi(r) = \dfrac{r}{\sqrt{g}} \sinh\left(\dfrac{\sqrt{g}}{r} \right) \underset{r\to\infty}{=} 1 + \cdots \end{cases}. \tag{51}$$

(e) *Logarithmic potentials* [6, 15]. We consider here the simplest case and any angular momentum $\ell \geqslant 0$:

$$\begin{cases} V(r) = \left[\dfrac{\alpha}{r^2 \,\mathrm{Log}\left(\frac{1}{r}\right)} + \dfrac{g}{r^2 \,\mathrm{Log}^2\left(\frac{1}{r}\right)} \right] \theta(R - r), \quad \alpha > 0, \quad R < 1, \\ \varphi(r) = r^{1/2} \left[\dfrac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2} - \nu - \mu\right)} M_{\mu,\nu}(x) + \dfrac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2} + \nu - \mu\right)} M_{\mu,-\nu}(x) \right], \\ \chi(r) = r^{1/2} [\alpha M_{\mu,\nu}(x) + \beta M_{\mu,-\nu}(x)], \end{cases} \tag{52}$$

where $M_{\mu,\nu}$ are Whittaker functions [13],

$$x = (2\ell + 1)\,\mathrm{Log}\,\frac{1}{r}, \qquad k = \frac{-\alpha}{2\ell + 1}, \qquad \nu = \mathrm{i}\sqrt{g - \frac{1}{4}}, \tag{53}$$

and α and β are determined according to (13) for $r \to 0$ and $r \to \infty$. Note here that, for $r \geqslant R$, we have the free equation (no potential), and, therefore, we must first adjust the free solution to the interior solution at $r = R$, as usual.

Note here that the singular part of the potential is just the first potential, and that is why we must choose $\alpha > 0$. The second potential is regular since it satisfies $rV(r) \in L^1(0)$. We can therefore choose $g \gtrless 0$. There are several more examples of singular potentials for which the radial equation can be solved explicitly. We refer the reader for details to [6].

(f) *Coulomb potential*. The Coulomb potential is regular at the origin, and so we have the usual solutions φ and χ, as defined previously. We choose, of course, the repulsive case.

The solutions can be read off from (49) adapted to $n < 2$, or else, be obtained in the standard way [1, 4]. One has, for $\ell = 0$

$$\begin{cases} V = \dfrac{\alpha}{r}, \quad \alpha > 0, \quad \ell = 0, \\ \varphi(r) = \sqrt{\dfrac{r}{\alpha}} I_1(2\sqrt{\alpha r}), \\ \chi(r) = -\pi\sqrt{\alpha r} K_1(2\sqrt{\alpha r}), \end{cases} \tag{54}$$

where I_1 and K_1 are the modified Bessel and Hankel functions, and similar formulae for $\ell > 0$. As we see here, the long-range tail of the potential leads to the exponential growth of φ at $r \to \infty, \varphi \sim r^{1/4}\exp(2\sqrt{\alpha r})$, and the exponential decrease of $\chi(r) \sim r^{1/4}\exp(-2\sqrt{\alpha r})$, to zero. This does not affect the validity of the change of the variable $r \to x = \varphi/\chi$, etc, of section 2, except that now x grows exponentially as $r \to \infty$. Note that I_1 and K_1 do not vanish, I_1 for $r > 0$, and K_1 for $r < \infty$ [13]. φ is again an increasing convex function, and χ a decreasing convex function. The only difference with the short-range potentials is that, now, $\chi^{-4}(r)$ grows exponentially, so that, in (37) and (41), if V_0 is chosen to be the Coulomb potential, the second potential may seem to become infinite as $r \to \infty$ $(x \to \infty)$. However, we have always assumed $V_1(r)$ to be short range, i.e. decreasing fast enough at infinity. It follows that $\chi_0^{-4}(r)V_1(x)$ is again short range. For instance, if $V_1(r) \sim r^{-4}$, then $\chi_0^{-4}V_1(r) \sim 1/(x^2 \operatorname{Log}^2 x)$, etc, i.e. $x\chi_0^{-4}V_1$ is L^1 in x at $x = \infty$.

Other long-range potentials of the form g/r^n, $n < 2$, can be dealt with in the same way by adapting (49) to $n < 2$, and one reaches similar conclusions as for the Coulomb potential. As for confining potentials such as the harmonic oscillator, etc, we shall consider them in a separate paper.

4. Concluding remarks

So far, all the potentials for which the radial Schrödinger equation has been shown to be soluble analytically in a closed form have their solutions given by various hypergeometric functions in appropriate variables [1, 3, 4, 6]. In fact, in many instances, as we have seen on examples, the hypergeometric functions simplify to the Bessel and Hankel functions of real or imaginary arguments. The only exceptions are Bargmann potentials [1, 3], for which the solutions are given in terms of rational functions of $\sinh\alpha_j r$, $\cosh\alpha_j r$, $\sin\beta_j r$, and $\cos\beta_j r$, $j = 1, \ldots, n$, where α_j and β_j are given by the positions of the poles and zeros of the S-matrix, and the r^{-4} potential [2], for which the solution is given in terms of Mathieu functions.

In the present paper, as seen in (37) and (41), the potentials themselves have their arguments given by ratios of hypergeometric functions, and the solutions are then hypergeometric functions of ratios of hypergeometric functions, as seen in (39). And this process can be repeated indefinitely, as we saw before. One may then ask what the potentials and their wavefunctions become in the limit. Also, we saw that, for both V_0 and V_1, we can take potentials which are very singular, but repulsive at the origin, like gr^{-n} and λr^{-m}, $g > 0, \lambda > 0, m, n > 2$. All kinds of combinations are therefore possible for V_0 and V_1.

It is trivial to construct infinitely many potentials for which the Schrödinger equation could be solved at zero energy. One can choose any positive, convex, and twice continuously differentiable function, which is decreasing, and such that

$$\chi_0(0) = 1, \qquad \chi_0(\infty) = \frac{1}{A}, \qquad 1 < A < \infty, \tag{55}$$

and write

$$V_0(r) = \frac{\chi_0''(r)}{\chi_0(r)}. \tag{56}$$

If $\chi_0(r)$ is decreasing fast enough to A^{-1} at infinity, then $V_0(r)$ is short range. $\varphi_0(r)$ is defined here by (14). For example:

$$\begin{cases} \chi_0(r) = \dfrac{1}{(1+\alpha)}\left[1 + \dfrac{\alpha}{(1+\beta r)^n}\right], \\ \alpha > 0, \quad \beta > 0, \quad n > 1. \end{cases} \tag{57}$$

According to (56), we have

$$V_0(r) = \frac{\alpha n(n+1)}{(1+\alpha)(1+\beta r)^{n+2}\chi_0(r)} \underset{r\to\infty}{\sim} r^{-n-2}. \tag{58}$$

However, all this is valid for this particular $V_0(r)$. If we try to introduce a coupling constant λ in front of $V_0(r)$, i.e. try to solve

$$\chi''(r) = \lambda V_0(r)\chi(r), \tag{59}$$

we usually do not find explicit solutions. This is indeed the case here.

Next, consider

$$\chi_0(r) = \frac{1 + \mathrm{e}^{-\mu r}}{2}, \qquad \mu > 0, \tag{60}$$

where $A = 2$. This leads to

$$V_0(r) = \frac{\chi_0''(r)}{\chi_0(r)} = \frac{\mu^2\,\mathrm{e}^{-\mu r}}{1 + \mathrm{e}^{-\mu r}}. \tag{61}$$

$\varphi_0(r)$ is then easily calculated from (14). In this case, one can solve (59) for any λ since (61) is a particular case of (4), and the solutions are given, in general, in terms of hypergeometric functions [4]. Only for $\lambda = 1$, they reduce to the simple form (60) for χ_0, and the corresponding $\varphi_0(r)$. The potentials we consider are those for which we have explicit solutions for all λ.

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Appendix

We begin by studying the properties of the solutions $\varphi_0(r)$, and $\chi_0(r)$ defined by (11), of the radial equation at $k = 0$ and $\ell = 0$:

$$\begin{cases} \varphi_0''(r) = V_0(r)\varphi_0(r), \\ \varphi_0(0) = 0, \quad \chi_0(0) = 1, \end{cases} \tag{A.1}$$

where $V_0(r)$ satisfies the assumptions shown in (8), for $\ell = 0$. Since the radial equation is homogeneous, we can normalize its solution φ as we wish. For (A.1), the usual convention is to put

$$\varphi_0'(0) = 1. \tag{A.2}$$

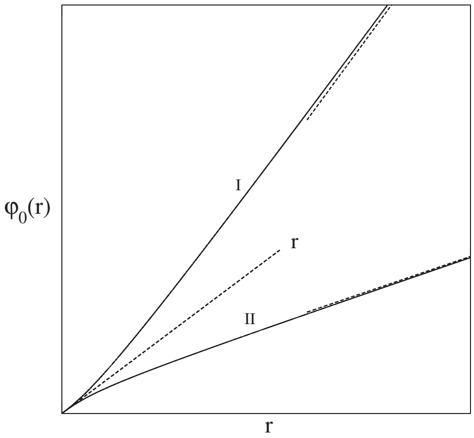

Figure 1. *I*, $V_0 > 0$; *II*, $V_0 < 0$, no bound states.

Then we can combine (A.1) and (A.2) into a single Volterra integral equation [1, 3, 4]

$$\varphi_0(r) = r + \int_0^r (r - r')V_0(r')\varphi_0(r')\,\mathrm{d}r'. \tag{A.3}$$

It can then be shown that, iterating the above equation, and using the assumptions on $V_0(r)$, namely that rV is L^1 at $r = 0$, $r^2V(r)$ is L^1 at $r = \infty$, one gets an absolutely and uniformly convergent series defining the solution $\varphi_0(r)$, together with the bound [1, 3, 4]

$$|\varphi_0(r)| \leqslant r\,\mathrm{e}^{\int_0^r r'|V_0(r')|\mathrm{d}r'} < Cr, \tag{A.4}$$

where C is an absolute constant less than $\exp\int_0^\infty r|V_0(r)|\,\mathrm{d}r$. Using this bound in (A.3), we find that indeed, for $r \to 0$, we have

$$\varphi_0(0) = 0, \qquad \varphi_0'(0) = 1, \tag{A.5}$$

and for $r \to \infty$,

$$\varphi_0(r) = r\left[1 + \int_0^\infty V_0(r')\varphi_0(r')\,\mathrm{d}r'\right] - \int_0^\infty r'V_0(r')\varphi_0(r')\,\mathrm{d}r' + o(1), \tag{A.6}$$

where all integrals are absolutely convergent.

There are now two cases:

(i) The potential $V_0(r)$ is positive. Then it is obvious on the iterated series of (A.3) that all the terms are positive, and so is $\varphi_0(r)$ for all r. It follows then from (A.1) that $\varphi_0(r)$ is a positive convex function of r. It increases indefinitely, and we have, on the basis of (A.6)

$$\begin{cases} \varphi_0(r) > 0, & \text{and convex,} \\ \varphi_0(r) = Ar + B + o(1), & \text{as} \quad r \to \infty, \quad 1 < A < \infty, \quad B < 0. \end{cases} \tag{A.7}$$

A schematic picture of φ_0 is shown in figure 1.

(ii) $V_0(r) < 0$, but not strong enough to have bound states. Then we find from the nodal theorem relating the bound states of $V_0(r)$ to the zeros (nodes) of $\varphi_0(r)$ for $r > 0$ [7], that $\varphi_0(r)$ is again positive, and since V_0 is now negative, φ_0 is positive and concave. A schematic picture of φ_0 is shown in figure 1. From (A.6), (A.7) is now replaced by

$$\begin{cases} \varphi_0(r) > 0, & \text{and concave,} \\ \varphi_0(r) = Ar + B + o(1), & \text{as} \quad r \to \infty, 0 < A < 1, B > 0. \end{cases} \tag{A.8}$$

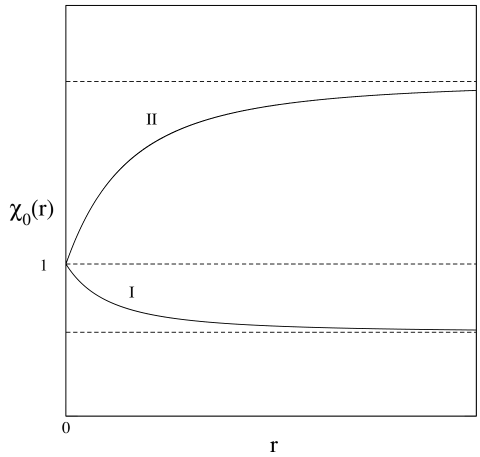

Figure 2. *I*, $V_0 > 0$; *II*, $V_0 < 0$, no bound states.

Remark. Here, if $A < 0$, then, since $\varphi_0'(0) = 1$, and $\varphi_0'(\infty) < 0$, φ_0 must have a zero in between, and therefore one has a bound state, in contradiction with our assumption of no bound states. If $A = 0$, this means that one is at the threshold of having a bound state. More precisely, that one has a resonance at zero energy [1, 3, 4], a possibility we have excluded also.

We now come to the second, independent solution $\chi_0(r)$, defined by (13). First of all, since $\varphi_0(r)$ is always positive for $r > 0$, and from (A.7) and (A.8), the integral is absolutely convergent at its upper limit, and so χ_0 is twice differentiable, and satisfies the same equation as φ_0. It is trivial to show that the Wronskian of the two, (11), is 1. Now, when $r \to 0$, the integral in (13) diverges, but since $\varphi_0(r) = r + o(r)$ as $r \to 0$, and there is $\varphi_0(r)$ in front of the integral, it is trivial to show that we have $\chi_0(0) = 1$, as shown in (11). Also, on the basis of the Wronskian (11), and (A.5), we find

$$\lim_{r\to 0} r\chi_0'(r) = 0. \tag{A.9}$$

This general property is a consequence of $rV_0(r) \in L^1$ at the origin. The derivative of $\chi_0(r)$ at $r = 0$ may be finite or infinite, depending on the behaviour of $V_0(r)$ near $r = 0$. If $V_0(r)$ itself is L^1 at $r = 0$, one can also write the integral equation [1, 3, 4]

$$\chi_0(r) = 1 + \int_0^r (r - r')V_0(r')\chi_0(r')\,\mathrm{d}r', \tag{A.10}$$

and iterate it, as we did with (A.3) for φ_0, to find the solution, which turns out now to be bounded everywhere. One then immediately sees on (A.10) that $\chi_0'(0)$ is finite. We have, therefore, according to (13), and (A.7) or (A.8) (see figure 2)

$$\begin{cases} \chi_0(0) = 1, \quad \chi_0(r) > 0 \quad \text{for all } r, \\ \chi_0(r) \text{ is a convex and decreasing function when } V_0 > 0, \\ \chi_0(r) \text{ is a concave and increasing function when } V_0 < 0, \\ \text{with no bound states,} \\ \chi_0(\infty) = \dfrac{1}{A} \neq 0, \infty; \quad V_0 > 0 \Rightarrow A > 1, \quad V_0 < 0 \Rightarrow A < 1. \end{cases} \tag{A.11}$$

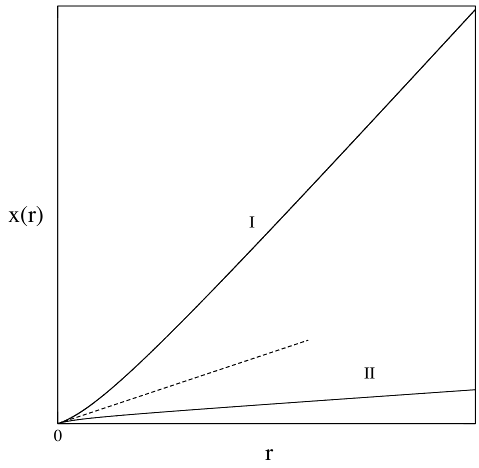

Figure 3. *I*, $V_0 \geqslant 0$; *II*, $V_0 < 0$, no bound states.

Consider now the mapping:

$$r \to x(r) = \frac{\varphi_0(r)}{\chi_0(r)}. \tag{A.12}$$

According to (A.11), this a perfectly regular and differentiable mapping, and is one to one since, according to (11), we have

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\varphi_0'\chi_0 - \varphi_0\chi_0'}{\chi_0^2(r)} = \frac{1}{\chi_0^2(r)} > 0. \tag{A.13}$$

It follows then, since $\varphi_0(r \to \infty) \to \infty$, and $\chi_0(r \to \infty) \to \frac{1}{A} \neq 0, \infty$, that the mapping is one to one:

$$r \in [0, \infty) \Leftrightarrow x \in [0, \infty), \quad x(0) = 0, \quad x(\infty) = \infty. \tag{A.14}$$

In fact, this mapping is twice continuously differentiable since

$$\frac{\mathrm{d}^2x}{\mathrm{d}r^2} = \frac{-2\chi_0'(r)}{\chi_0^3(r)}, \tag{A.15}$$

and $\chi_0'(r)$ is a continuous function of r for $r \geqslant 0$. This last property follows from $\chi_0''(r) = V_0(r)\chi_0(r)$, where, by assumption, $V_0(r) \in L^1$ for $r > 0$. Since $\chi_0(r)$ is a continuous function, $\chi_0''(r)$ is also L^1 for all $r > 0$, and so $\chi_0'(r)$ is continuous for $r > 0$. χ_0' cannot have jumps [14], see figure 3.

Higher waves, $\ell > 0$. So far, we have been assuming $\ell = 0$. It is easy to extend the results to the case $\ell > 0$ in (38), i.e. to begin with

$$\varphi_0''(r) = \left[V_0(r) + \frac{\ell(\ell+1)}{r^2}\right]\varphi_0(r), \tag{A.16}$$

and then add the potential V_1 in (37). V_0 is, as before, supposed to be such that $rV_0 \in L^1$ at $r = 0$, and $r^2V_0 \in L^1$ at $r = \infty$. We shall see later that we need more rapid decrease at infinity. The regular solution φ_0 is usually normalized as follows:

$$\varphi_0(r) = \frac{r^{2\ell+1}}{(2\ell+1)!!} + o(r^{\ell+1}), \qquad r \to 0. \tag{A.17}$$

One can then combine (A.16) and (A.17) into the single Volterra integral equation

$$\varphi_0(r) = \frac{r^{\ell+1}}{(2\ell+1)!!} + \int_0^r \frac{r^{2\ell+1} - r'^{2\ell+1}}{(2\ell+1)r^\ell r'^\ell} V_0(r')\varphi_0(r')\,\mathrm{d}r'. \tag{A.18}$$

Solving this equation by iteration, one finds again, as for the case $\ell = 0$, an absolutely and uniformly convergent series defining the solution φ_0, together with a bound similar to (A.4) for all finite $r \geqslant 0$:

$$|\varphi_0(r)| \leqslant C r^{\ell+1} \exp\left(\int_0^r r'|V_0(r)|\,\mathrm{d}r'\right) \leqslant C' r^{\ell+1}, \tag{A.19}$$

where C and C' are absolute finite constants [1, 3, 4]. Using (A.19) into (A.18), one sees immediately that, for $r \to 0$,

$$\varphi_0(r) = \frac{r^{\ell+1}}{(2\ell+1)!!} + o(r^{2\ell+1}), \qquad \varphi_0'(r) = \frac{r^{2\ell}}{(2\ell-1)!!} + o(r^{2\ell}). \tag{A.20}$$

For all the above results, we need only $rV_0 \in L^1(0)$. Also, by assumption, there are no bound states for (A.16). It follows again that, by the nodal theorem [7], $\varphi_0(r)$ cannot vanish for $r > 0$. Because of (A.20), we find therefore that

$$\varphi_0(r) > 0 \quad \text{for all } r > 0. \tag{A.21}$$

From this, and (A.16), it follows immediately that if $V_0(r) > 0$, then $\varphi_0(r)$ is convex. For $V_0(r) < 0$, the situation is more subtle than for the case of $\ell = 0$, and φ_0 may become concave in some interval (R_1, R_2).

We wish now to look at the behaviour of $\varphi_0(r)$ as $r \to \infty$. We assume here

$$r^{2\ell+2} V_0(r) \in L^1(\infty). \tag{A.22}$$

Then we can let $r \to \infty$ in (A.18), to find

$$\begin{aligned}\varphi_0 \underset{r\to\infty}{=} &\left[\frac{1}{(2\ell+1)!!} + \int_0^\infty \frac{1}{2\ell+1} r'^{-\ell} V_0(r')\varphi_0(r')\,\mathrm{d}r'\right] r^{\ell+1} \\ &- \frac{1}{2\ell+1}\left[\int_0^\infty r'^{\ell+1} V_0(r')\varphi_0(r')\,\mathrm{d}r'\right] r^{-\ell} + \cdots \\ &= A_\ell r^{\ell+1} + B_\ell r^{-\ell} + \cdots.\end{aligned} \tag{A.23}$$

Since $\varphi_0(r)$ never vanishes (absence of bound states), A_ℓ must be always positive. If $V_0 > 0$, then $A_\ell > 1$, and if $V_0 < 0$, $0 < A_\ell < 1$. For B_ℓ, it is just the opposite. These are quite the same as for $\ell = 0$.

We can now define the second (independent) solution $\chi_0(r)$ by (13) again, and we find now, using (A.20) and (A.23), that

$$\begin{cases} \chi_0(r) = \dfrac{(2\ell-1)!!}{r^\ell} + \cdots, & r \to 0 \\ \chi_0(r) = \dfrac{1}{(2\ell+1)A_\ell} r^{-\ell}, & r \to \infty. \end{cases} \tag{A.24}$$

If we introduce the same variable $x = x(r)$ as before

$$x = x(r) = \frac{\varphi_0(r)}{\chi_0(r)}, \tag{A.25}$$

we find

$$\begin{cases} x(r) = Ar^{2\ell+1} + \cdots, & r \to 0 \\ x(r) = Br^{2\ell+1}, & r \to \infty. \end{cases} \tag{A.26}$$

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